Positive and non-positive solutions
for an inviscid dyadic model.
Well-posedness and regularity.

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Abstract

We improve regolarity and uniqueness results from the literature for the inviscid dyadic model. We show that positive dyadic is globally well-posed for every rate of growth β of the scaling coefficients $k_n = 2^{\beta n}$. Some regularity results are proved for positive solutions, namely $\sup_n n^{-\alpha} k_n^{\frac{1}{3}} X_n(t) < \infty$ for a.e. t and $\sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} X_n(t) \leq Ct^{-1/3}$ for all t. Moreover it is shown that under very general hypothesis, solutions become positive after a finite time.

Introduction

Well-posedness and regularity for Navier-Stokes and Euler equations represent a major point of interest in mathematics. The study of estimates of the nonlinear term $(u \cdot \triangledown)u$ in particular is important, since this term is associated with the so-called cascade of energy from lower to higher modes.

A very rough idea of this phenomenon is the following. Fix a time t and decompose the velocity u on the frequencies $u = \sum_k u_k e_k$, where e_k are "wave functions" (for example $e_k = \sin(\langle k, \cdot \rangle)$ or $e_k = e^{i\langle k, \cdot \rangle}$) and u_k are the corresponding coefficients (with $\sum_k |u_k|^2 = \|u\|_{L^2}^2$). Then the regularity of u can be associated to how fast the coefficients u_k go to zero as $|k|$ tends to infinity. More precisely, u has N-th derivative in L^2 if $\sum_k(|k|^N|u_k|)^2 < \infty$. The bilinear term $(u \cdot \triangledown)u$ acts on the dynamics of the coefficients u_k by mixing different components, in that if h and k are two active frequencies $(u_h, u_k \neq 0)$, then the term $(u \cdot \triangledown)u$ will activate the component $h + k$ and so on, activating higher and higher frequencies in a phenomenon called *energy cascade* [6].

In this paper we study the inviscid dyadic model, which is a shell-type model of Euler equations which was early introduced in [9] and then again in recent times in [11] and [13]. This model, represented in equations (1) below, exploits some of the properties of Euler equations (see among the others [6]

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and [10]) on a much simpler structure. Informal derivations of this model from Euler equations are given in [9], [11] and [7]; from an intuitive point of view, one should imagine that the variable X_n in (1) represents a global coefficient for all components u_k with $|k|$ of order 2^n. These equations, like Euler's, are homogeneous of degree 2, they are formally conservative and moreover they show the energy cascade phenomenon, which in this setting is very clearly understood [4].

To collect previous results on the dyadic model, one should immediately distinguish whether the initial condition has all positive components or not, since the dynamics of energy cascade is strongly dependent on the sign of X_n. If all X_n's are positive, energy moves from lower modes to higher ones. If all X_n's are negative, energy moves from higher modes to lower ones. In [4] and [8] it is shown that in the case of positive components, energy moves to higher modes faster and faster, in such a way that a positive fraction of energy gets lost "at infinity" in finite time and one can show that in this case the energy $\|X(t)\|_{l^2}^2$ goes to zero like t^{-2}. On the other hand in [8] it is proved that if a positive forcing term is included in the model (which puts energy into the first component), the energy converges exponentially fast to a fixed value (corresponding to the stationary solution). In both cases the model is not really conservative in the end, since energy moves to infinity and there it disappears: this phenomenon is called anomalous dissipation.

On the other hand, if all components are negative, the opposite situation can occur: energy can enter from "infinity" into the system. In [4] explicit solutions are costructed in which there is an anomalous increase of energy, immediately yielding non-uniqueness of solutions for the negative dyadic.

In this paper we prove that the positive dyadic is globally well-posed, extending the uniqueness result in [1] to arbitrary rate of growth of the coefficients k_n in system (1). This means in particular that the escape of energy at infinity does never preclude well-posedness; on the contrary, in the negative dyadic, the input of energy from infinity immediately destroys well-posedness.

It is interesting to confront this with the stochastic dyadic model introduced in [2], where the distinction between positive and negative solutions is meaningless, since noise causes infinite sign changes in every time interval. In [3] and [2] it is proved that there is escape of energy at infinity and (weak) uniqueness of solutions, so also in this case energy cannot enter from infinity and the problem is well-posed.

Section 2 deals with the connection between negative and positive dyadic. It is shown that under minimal hypothesis all components become positive in finite time and stay positive forever. Based on this fact all the rest of the paper is restricted to positive initial conditions.

Sections 3, 4 and 5 deal with uniqueness and regularity and are very much interlinked. Uniqueness was already proved in [1] for $k_n = 2^{\beta n}$ and $\beta \leq 1$ and in [5] for any β in a class of regular enough solutions. It is now

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extended by Theorem 13 to arbitrary β and l^2 initial condition.

As already stated we are interpreting the components of the solution as something similar to Fourier coefficients, so regularity means smallness of components X_n for n large. The vague idea is that X_n tends to zero as $k_n^{-1/3}$. The first results in the literature are of lack of regularity. In [7], [11], [8], [12] and [4] it is shown that if the initial condition is in l^2, then all solutions are Leray-Hopf and nontheless a blow-up occurs in finite time, in the sense that for all $\epsilon > 0$ the quantities

$$\sup_n k_n^{\frac{1}{3}+\epsilon} X_n(t), \qquad \sum_n k_n^{\frac{1}{3}} X_n(t), \qquad \sup_n \int_0^t k_n^{(1+\epsilon)} X_n^3(s) ds,$$

become infinite in finite time, even if they were finite for $t = 0$.

On the other hand one first important regularity result can be found again in [8], where for $\beta < 3$, the authors prove that for all $\epsilon > 0$,

$$\sup_n \int_0^t (k_n^{1/3-\epsilon} X_n(s))^2 ds < +\infty.$$

Our main results on regularity are Theorem 10, Lemma 14, Theorem 15 and Theorem 17 which, through some corollaries, imply that

1. $\sup_n n^{-\alpha} k_n^{\frac{1}{3}} X_n(t) < \infty$ for all $\alpha > \frac{1}{3}$ and for a.e. $t > 0$;

2. $\sum_n n^{-\alpha} k_n^{\frac{1}{3}} X_n(t) < \infty$ for all $\alpha > \frac{4}{3}$ and for a.e. $t > 0$;

3. $\sup_n \int_0^t n^{-1} k_n X_n^3(s) ds < \infty$ for all $t > 0$;

and moreover

4. $\sup_n k_n^{\frac{1}{3}-\frac{1}{3\beta}} X_n(t) \le C t^{-1/3}$ for all $t > 0$.

1   Model

A very natural space for the dynamics of the dyadic is $H := l^2(\mathbb{R})$, the Hilbert space of square-summable sequences with the usual norm which we will denote simply by $\| \cdot \|$.

Let $\beta > 0$ and $x = (x_n)_{n \ge 1} \in H$. Consider the following Cauchy problem

$$\begin{cases} \dot{X}_n = k_{n-1} X_{n-1}^2 - k_n X_n X_{n+1}, \\ X_n(0) = x_n, \end{cases} \qquad n \ge 1 \quad t \ge 0 \qquad (1)$$

where $X_0 = 0$, $k_0 = 0$ and $k_n = 2^{\beta n}$ for $n \ge 1$.

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Definition 1. *A weak solution is a sequence $X = (X_n)_{n \geq 1}$ of differentiable functions on all $[0, \infty)$, satisfying (1).*

A finite energy solution is a weak solution such that $X(t)$ is in H for all $t \geq 0$.

The following proposition (whose proof is immediate) shows that without loss of generality we can suppose that the initial condition x has positive first component $x_1 > 0$ and arbitrarily small norm.

Proposition 2. *Suppose $(X_n)_{n \geq 1}$ is a weak solution of (1) with initial condition $x \neq 0$. We denote by $\bar{n}$ its first non-zero index: $\bar{n} := \min\{n \geq 1 : x_n \neq 0\}$, so that $X_n \equiv 0$ for $n < \bar{n}$, while $X_{\bar{n}}(t) \neq 0$ for all $t > 0$. Let $\alpha > 0$ and let, for $n \geq 1$,*

$$Y_n(t) := \begin{cases} X_n(t) & n \neq \bar{n} \\ -X_n(t) & n = \bar{n} \end{cases}, \qquad Z_n(t) := X_{n+\bar{n}-1}\left(\frac{t}{k_{\bar{n}-1}}\right), \qquad (2)$$

$$W_n(t) := \alpha X_n(\alpha t). \qquad (3)$$

Then all of the above, $(Y_n)_{n \geq 1}$, $(Z_n)_{n \geq 1}$ and $(W_n)_{n \geq 1}$, are weak solutions of (1) each one with its own initial condition.

Definition 3. *A Leray-Hopf solution is a finite energy solution such that $\|X(t)\|$ is a non-increasing function of t.*

We also introduce the notation for the finite-size blocks energy: for all $n \geq 1$ let

$$E_n(t) := \sum_{i \leq n} X_i^2(t). \qquad (4)$$

A direct computation shows that

$$E_n' = -2k_n X_n^2 X_{n+1}, \qquad (5)$$

so we can study the variations of energy by looking to the sign of components.

Proposition 4. *Let X be a weak solution of (1) and $t_0 \geq 0$. If $X_n(t_0) > 0$ then $X_n(t) > 0$ for all $t \geq t_0$. If $X_n(t_0) \geq 0$ then $X_n(t) \geq 0$ for all $t \geq t_0$.*

Proof. Simply apply the variation of constants formula to (1)

$$X_n(t) = X_n(t_0)e^{-\int_{t_0}^t k_n X_{n+1}(s)ds} + \int_{t_0}^t k_{n-1} X_{n-1}^2(s)e^{-\int_s^t k_n X_{n+1}(\tau)d\tau}ds.$$

The thesis follows. $\qquad \square$

Proposition 5. *If the initial condition of system (1) has infinitely-many non-negative components, then every solution is Leray-Hopf.*

Proof. Let $(n_i)_{i \geq 1}$ be an increasing sequence such that $x_{n_i} \geq 0$. By Proposition 4, $X_{n_i}(t) \geq 0$ for all $t \geq 0$, so that for all i, E_{n_i-1} is a non-increasing function. Since $E_{n_i-1} \uparrow \|X\|^2$ pointwise as $i \to \infty$, $\|X\|^2$ is also non-increasing. $\qquad \square$

We conclude this section by introducing the concept of a positive solution.

Definition 6. *We denote by H^+ the set of points in H with all positive components,*

$$H^+ = \{x \in H : x_i > 0 \text{ for all } i \geq 1\}.$$

A positive solution is any solution such that $X(t) \in H^+$ for all $t \geq 0$. Of course positive solutions are always Leray-Hopf.

It is a consequence of Proposition 4 that H^+ is closed for the dynamics, and actually a slighty stronger result was proved in [4]: it is enough for the initial condition to have a positive first component and all the other components non-negative, to prove that $X(t) \in H^+$ for all $t > 0$.

Positive solutions will turn out to have interesting regularizing properties, in the next section we will prove that under very general hypothesis, all solutions become positive after a finite time.

2 Negative components

The following statement shows that under very general hypothesis, all solutions become positive after a finite time.

Only for this section we will weaken the hypothesis that the initial condition belongs to H.

Theorem 7. *Let $x \in \mathbb{R}^{\mathbb{N}+}$ be any initial condition with $x_1 > 0$ and let X be a weak solution. We suppose that one of the following hypothesis hold:*

1. *$x \in l^\infty$; there is an increasing sequence of indices $(n_i)_{i \geq 1}$ such that $x_{n_i} \geq 0$ for all $i \geq 1$; for some $\delta \in (0,1)$, some $j_0 \in \mathbb{N}$ and for all $j \geq j_0$, $n_{j+1} \leq k_{n_j}^{2/3\delta}$.*

2. *$x \in H$ and X is a Leray-Hopf solution.*

Then there exists $\tau > 0$ such that for all $t \geq \tau$ we have $X(t) \in H^+$.

We will need the Lemmas 8 and 9 below. Both of them use the following quantity,

$$\eta_n := \sum_{k \leq \omega_n} x_k^2 = E_{\omega_n}(0),$$

where $\omega_n = \inf\{i \geq n : x_{i+1} \geq 0\}$. This quantity is useful to bound X_n and can be easily bounded itself, as we show presently.

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Depending on which one of the two hypothesis of the theorem holds, it may be either that $\omega_n < \infty$ and $x_{\omega_n+1} \geq 0$, or $\omega_n = \infty$ and $\eta_n = \|x\|^2$. By applying (5) or the Leray-Hopf property, in both cases, for all $n \geq 1$ and $t \geq 0$,

$$X_n^2(t) \leq E_{\omega_n}(t) \leq \eta_n. \tag{6}$$

In the case of Leray-Hopf solutions, $\eta_n = \|x\|^2$ for all n. In the case of l^∞ initial condition, $\eta_n \leq \omega_n \|x\|_{l^\infty}^2$ and moreover, if i is such that $n_i \leq n < n_{i+1}$, then $\omega_n = n_{i+1} - 1$, so

$$\frac{\eta_n}{\|x\|_{l^\infty}^2} \leq \omega_n \leq k_{n_i}^{2/3\delta} \leq k_n^{2/3\delta}, \tag{7}$$

definitively.

Lemma 8. *In the same hypothesis of Theorem 7, if $X_n(t) \geq a > 0$, then $X_{n+1}(t + v_n(a)) \geq 0$, where*

$$v_n(a) = \frac{2^{2\beta}\eta_n^2 + a^4}{k_{n+1}a^4\sqrt{\eta_n}}.$$

Moreover, for all $a > 0$

$$\sum_{n=1}^{\infty} v_n(a) < \infty. \tag{8}$$

Proof. For all $x \in \mathbb{R}$, let $\tau_x := \inf\{s \geq 0 : X_{n+1}(s) \geq x\} \in [0, \infty]$. If $X_{n+1}(t) \geq 0$ there is nothing to prove, so we suppose $\tau_0 > t$.

On $[t, \tau_0)$, $X_{n+1} \leq 0$, so $X_n' = k_{n-1}X_{n-1}^2 - k_n X_n X_{n+1} \geq 0$ and hence $X_n \geq a$ on the same interval.

Let x be any positive number (it will be fixed a few lines below). We want to prove that

$$\tau_{-x} \leq t + \frac{\eta_n}{2k_n a^2 x}. \tag{9}$$

If $X_{n+1}(t) > -x$, $\tau_{-x} \leq t$ and we are done. On the other hand, if $\tau_{-x} > t$, on $[t, \tau_{-x})$, $X_{n+1} \leq -x$ and the energy of components from $n+1$ to ω_n decreases at least linearly, as we show presently. By (5), for all $s \in [t, \tau_{-x})$,

$$E_n(s) = -\int_t^s 2k_n X_n^2(u) X_{n+1}(u) du \geq 2(s-t)k_n a^2 x,$$

so that by (6), we obtain bound (9):

$$0 \leq \sum_{i=n+1}^{\omega_n} X_i^2(s) = E_{\omega_n}(s) - E_n(s) \leq \eta_n - 2k_n a^2 x(s-t).$$

Now let $x = \frac{k_n a^2}{2k_{n+1}\sqrt{\eta_n}}$. We claim that with this choice of x, $X_{n+1}' \geq k_n a^2/2$ on $[\tau_{-x}, \tau_0)$, yielding

$$\tau_0 - \tau_{-x} \leq \frac{2x}{k_n a^2} = \frac{1}{k_{n+1}\sqrt{\eta_n}}. \tag{10}$$

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We prove the claim,

$$X'_{n+1} = k_n X_n^2 - k_{n+1} X_{n+1} X_{n+2} \geq k_n a^2 - k_{n+1} X_{n+1} X_{n+2}.$$

Since $X_{n+1} \leq 0$, if $x_{n+2} \geq 0$ we conclude immediately that $X'_{n+1} \geq k_n a^2$. On the other hand, if $x_{n+2} < 0$, then $\omega_n \geq n + 2$ and by (6) $X_{n+2} \geq -\sqrt{\eta_n}$ so that

$$X'_{n+1} \geq k_n a^2 + k_{n+1} \sqrt{\eta_n} X_{n+1}.$$

Since $X_{n+1}(\tau_{-x}) = -x$ and the RHS becomes negative only for $X_{n+1} < -2x$, we see that $X_{n+1} \geq -x$ on $[\tau_{-x}, \tau_0)$ and finally

$$X'_{n+1} \geq k_n a^2 - k_{n+1} \sqrt{\eta_n} x = k_n a^2 / 2.$$

Putting (9) and (10) together, we find $X_{n+1}(t + v_n(a)) \geq 0$ with

$$v_n(a) := \frac{\eta_n}{2 k_n a^2 x} + \frac{1}{k_{n+1} \sqrt{\eta_n}} = \frac{2^{2\beta} \eta_n^2 + a^4}{k_{n+1} a^4 \sqrt{\eta_n}},$$

so the first part of the statement is proved.

Finally, we turn to prove the convergence of (8). This is obvious in the case of Leray-Hopf solutions where η_n does not depend on n. In the other case we can reduce ourselves to prove $\sum_{n=1}^{\infty} \frac{\eta_n^{3/2}}{k_n} < \infty$, which follows from (7). $\qquad\square$

Lemma 9. *In the same hypothesis of Theorem 7, there exist two summable sequences of positive numbers $(a_n)_{n \geq 1}$, $(s_n)_{n \geq 1}$ depending only on x, such that for all $n \geq 1$, for all $t > 0$ and for all $\varepsilon \in (0, 1]$,*

$$\text{if} \quad X_n(t) \geq 0, X_{n+1}(t) \geq 0, \tag{11}$$

$$\text{then} \quad E_n(t + \varepsilon^{-2} s_n) - E_{n-1}(t) \leq \varepsilon a_n. \tag{12}$$

Proof. We follow the lines of the first part of Lemma 7 in [4], weakening the hypothesis of positivity.

For $n \geq 1$, let $s_n = \frac{2^{1+\beta}}{x_1} k_n^{-(1-\delta)/3}$, $a_n = C k_n^{-(1-\delta)/3}$ and

$$b_{n,\varepsilon} = 2 a_n^2 s_n k_{n-1} \left(1 + \frac{2}{k_{n+1} \sqrt{\eta_{n+2}} \varepsilon^{-2} s_n} \right)^{-1}.$$

By using $\varepsilon \leq 1$, $\eta_{n+2} \geq x_1^2$ and after some computations, one shows that

$$b_{n,\varepsilon} \geq k_{n+2}^{\delta} \left(1 + \frac{2}{k_{n+1} x_1 s_n} \right)^{-1} K' C^2 = k_{n+2}^{\delta} \left(1 + k_n^{-2/3 - \delta/3} \right)^{-1} K' C^2$$

$$\geq k_{n+2}^{\delta} \left(1 + k_1^{-2/3} \right)^{-1} K' C^2 \geq k_{n+2}^{\delta} K C^2 \geq K C^2,$$

where $K > 0$ does not depend on n or ε.

Thanks to this inequality, by a suitable choice of C, we can impose that $b_{n,\varepsilon} \geq \eta_{n+2}^{3/2}$, for all $n \geq 1$ and $\varepsilon \in (0, 1]$.

In fact, in the case of Leray-Hopf solutions we choose C such that $KC^2 \geq \|x\|^3$, so for all $n \geq 1$, $\inf_\varepsilon b_{n,\varepsilon} \geq \|x\|^3 = \eta_{n+2}^{3/2}$. In the case of l^∞ initial condition we choose C such that $KC^2 \geq \|x\|_{l^\infty}^3$, so for all $n \geq 1$, $\inf_\varepsilon b_{n,\varepsilon} \geq \|x\|_{l^\infty}^3 k_{n+2}^\delta$ and then we apply (7).

Let $h = \varepsilon^{-2} s_n$ and fix n and t. By (11), E_{n-1} and E_n are both nonincreasing in $[t, t + h]$, so

$$E_n(t + \varepsilon^{-2}s_n) - E_{n-1}(t) \leq E_n(t+s) - E_{n-1}(t+s) = X_n^2(t+s),$$

for all $s \in [0, h]$.

Now we proceed by contradiction. Suppose that (12) does not hold. Then the above inequality implies that $X_n^2 > \varepsilon a_n$ on $[t, t + h]$, yielding

$$
\begin{aligned}
E_n(t + \varepsilon^{-2}s_n) - E_{n-1}(t) &= \int_0^h E_n'(t+s)ds + X_n^2(t) \\
&= -\int_0^h 2k_n X_n^2(t+s)X_{n+1}(t+s)ds + X_n^2(t) \\
&\leq -2\varepsilon a_n k_n \int_0^h X_{n+1}(t+s)ds + X_n^2(t). \quad (13)
\end{aligned}
$$

We turn our attention to X_{n+1}. On the interval $[t, t + h]$ we have

$$X_{n+1}' = k_n X_n^2 - k_{n+1}X_{n+1}X_{n+2} \geq \varepsilon a_n k_n - k_{n+1}\sqrt{\eta_{n+2}}X_{n+1}.$$

We get

$$X_{n+1}(t+s) \geq \varepsilon a_n k_n \int_0^s e^{k_{n+1}\sqrt{\eta_{n+2}}(\tau-s)}d\tau.$$

If this integral is computed and substituted into (13), using the inequality $e^{-x} - 1 + x \geq x\left(1 + \frac{2}{x}\right)^{-1}$ one gets

$$E_n(t + \varepsilon^{-2}s_n) - E_{n-1}(t) \leq X_n^2(t) - \frac{2\varepsilon^2 a_n^2 k_n^2}{k_{n+1}\sqrt{\eta_{n+2}}}\frac{h}{1 + \frac{2}{k_{n+1}\sqrt{\eta_{n+2}}h}}$$

$$= X_n^2(t) - \frac{2a_n^2 s_n k_{n-1}\eta_{n+2}^{-1/2}}{1 + \frac{2}{k_{n+1}\sqrt{\eta_{n+2}}\varepsilon^{-2}s_n}} = X_n^2(t) - b_{n,\varepsilon}\eta_{n+2}^{-1/2} \leq X_n^2(t) - \eta_{n+2} \leq 0.$$

Since we were pretending (12) would not hold, this is a contradiction. □

Proof of Theorem 7. Let $(a_n)_{n \geq 1}$ and $(s_n)_{n \geq 1}$ be as in Lemma 9 and let

$$\gamma := \sqrt{x_1^2 - \varepsilon \sum_{i=1}^\infty a_i},$$

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with $\varepsilon > 0$ small enough to make the radicand positive. Let $t_0 = 0$. By Lemma 8 there exist a time t_1 such that $X_2(t_1) \geq 0$. For $n \geq 1$ let

$$t_{n+1} = t_n + \varepsilon^{-2}s_n + v_{n+1}(\gamma),$$

where v_n is given again by Lemma 8. Notice that $\sum_{k=1}^{\infty} s_k$ and $\sum_{k=1}^{\infty} v_k(\gamma)$ converge by the two lemmas, hence $t_k \uparrow \sup_k t_k =: \tau < \infty$.

We will show by induction that $X_n(t_{n-1}) \geq 0$ for all $n \geq 1$, and this will prove the positivity of all components at time τ.

The cases $n = 1, 2$ are already done. Now, let $n \geq 2$, we aim to prove that $X_{n+1}(t_n) \geq 0$. By inductive hypothesis, for $k = 1, 2, \ldots, n$ we know $X_k(t_{k-1}) \geq 0$, meaning in particular that $X_k(t) \geq 0$ and for all $t \geq t_{n-1}$. By Lemma 9 applied with $t = t_k$,

$$E_k(t_k + \varepsilon^{-2}s_k) - E_{k-1}(t_k) \leq \varepsilon a_k, \qquad k = 1, 2, \ldots, n-1.$$

Using the fact that when $X_{k+1} \geq 0$, E_k is nonincreasing we get

$$E_k(t_{k+1}) - E_{k-1}(t_k) \leq \varepsilon a_k, \qquad k = 1, 2, \ldots, n-1.$$

By summing over k the latter or the former inequalities, we obtain

$$E_{n-1}(t_{n-1} + \varepsilon^{-2}s_{n-1}) \leq \sum_{k=1}^{\infty} \varepsilon a_k, \tag{14}$$

yielding

$$X_n^2(t_n - v_n(\gamma)) \geq E_n(t_n - v_n(\gamma)) - \varepsilon \sum_{k=1}^{\infty} a_k.$$

Either $X_{n+1}(t_n) \geq 0$ (and we are done), or $X_{n+1} < 0$ and E_n is nondecreasing on $[0, t_n]$, giving

$$X_n^2(t_n - v_n(\gamma)) \geq E_n(0) - \varepsilon \sum_{k=1}^{\infty} a_k \geq x_1^2 - \varepsilon \sum_{k=1}^{\infty} a_k = \gamma^2.$$

X_n is nonnegative since t_{n-1}, so we can resolve the sign yielding $X_n(t_n - v_n(\gamma)) \geq \gamma$. Lemma 8 applies and we conclude $X_{n+1}(t_n) \geq 0$.

We still have to prove that $X(t) \in l^2$ for $t \geq \tau$, but this is an easy consequence of positivity: we know that for $t \geq t_{n-1}$, $X_n(t) \geq 0$ and hence $E'_{n-1}(t) \leq 0$, so, from inequality (14), we deduce

$$E_{n-1}(\tau) \leq E_{n-1}(t_{n-1} + \varepsilon^{-2}s_{n-1}) \leq \sum_{k=1}^{\infty} \varepsilon a_k.$$

Letting n go to infinity we get the result. $\qquad\square$

3 Regularity of positive solutions

When only solutions with all positive components are considered, we know from [4] that even if the initial condition is in l^∞, all solutions become l^2 immediately and are Leray-Hopf from that point onward.

Even if the solution is very regular at some time, energy can be conserved only for some finite time-interval and then anomalous dissipation starts. This phenomenon implies that some regularity was lost, in particular it means that for all $\epsilon > 0$ the quantities

$$\sup_n k_n^{\frac{1}{3}+\epsilon} X_n(t) \qquad \text{and} \qquad \sum_n k_n^{\frac{1}{3}} X_n(t),$$

become infinite in finite time, even if they were finite for $t = 0$.

The results of this and the following sections will prove instead that some regularity is still mantained, specifically,

1. for a.e. $t > 0$, and all $\alpha > \frac{1}{3}$, $\sup_n n^{-\alpha} k_n^{\frac{1}{3}} X_n(t) < \infty$;

2. for a.e. $t > 0$, and all $\alpha > \frac{4}{3}$, $\sum_n n^{-\alpha} k_n^{\frac{1}{3}} X_n(t) < \infty$;

3. $\sup_n k_n^{\frac{1}{3}-\frac{1}{3\beta}} X_n(t) \leq Ct^{-1/3}$ for all $t > 0$.

Statement 2 follows trivially from statement 1, which in turn is a consequence of Theorem 10, proved in Corollary 11. Statement 3 follows from uniqueness (Section 4) and Theorem 15, and is proved in Theorem 17.

Theorem 10. *Let $x = (x)_{n\geq 1} \in l^2$ with $x_n \geq 0$ for all n; let $X = (X)_{n\geq 1}$ be a weak solution of (1) with initial condition x. Then there exists a constant c depending only $||x||_{l^2}$ and β such that for any positive, non-increasing sequence $(a_n)_{n\geq 1}$ the following inequality holds*

$$\mathcal{L}\{t > 0 | X_n(t) > a_n \text{ for some } n\} \leq c \sum_n \frac{1}{k_n a_n^3}, \tag{15}$$

where $\mathcal{L}$ denotes the Lebesgue measure.

The quantity $c = 2^{7+\beta}||x||_{l^2}^2$ satisfies this theorem.

The sequences $(a_n)_{n\geq 1}$ which are meaningful for this statement are those for which the sum on the right-hand side is finite. The example which in particular is important for us is $a_n := Mk_n^{-\frac{1-\epsilon}{3}}$ with M large, for in that case one proves that the bound $X_n(t) \leq Mk_n^{-\frac{1-\epsilon}{3}}$ holds for all t except for a set of small measure $\frac{c'||x||^2}{M^3}$ and all $n \geq 1$.

Proof. We suppose that the sum $\sum_n \frac{1}{k_n a_n^3}$ is finite.

Let $I := \cup_n \{t \in (0, +\infty) | X_n(t) > a_n\}$. The set I is open, so it is possible to approximate it from inside with finite unions of intervals. To prove inequality (15) it will be sufficient to show that $\mathcal{L}(J) \leq c \sum_n \frac{1}{k_n a_n^3}$ for any possible set $J \subseteq I$ which is the finite union of intervals. Let J be the union of a finite number of disjoint intervals

$$J = \cup_{k=1}^m [b_k, c_k),$$

with $J \subseteq I$. Since we changed I with J, we enjoy the following property: for all $M > 0$, the set $J \setminus [0, M)$ either is empty, or it has a minimum.

Let $T := \sup(J) + \sup_n \{\frac{3}{k_n \cdot a_n}\}$ (a time large enough to be sure that definition (16) below always yields $t_i < T$).

We are going to define a family of intervals $[s_i, t_i)$ whose union covers J and such that their total measure is less than $c \sum_n \frac{1}{k_n a_n^3}$. To each interval we will associate one component n_i of the solution, in such a way that we can control X_{n_i} on the interval. The variables s_i, t_i and n_i will be defined by transfinite induction on t_i, starting from $t_0 = 0$. Notice that the definitions will ensure that $s_i \leq t_i \leq T$ for all ordinals i, with $s_i = t_i$ if and only if $s_i = t_i = T$. Moreover, if i and j are ordinals with $i < j$, then $t_i \leq s_j$.

We now give the definition by transfinite induction. Let i be an ordinal and suppose we already defined t_j for all $j < i$. Let $J_i := \{t \in J | t \geq t_j \ \forall j < i\}$, if J_i is empty we define $s_i = T$ and $t_i = T$, otherwise J_i has a minimum and we can define

$$s_i := \min(J_i),$$
$$n_i := \min\{n \geq 1 | X_n(s_i) > a_n, X_n(s_i) \geq X_{n+2}(s_i)\}.$$

The fact that n_i is well-defined follows by $s_i \in I$, $x \in l^2$ and $(a_n)_n$ non-increasing. The most subtle step in the proof is the definition of t_i below:

$$t_i := \min \left\{ \begin{array}{l} \inf\{t > s_i | X_{n_i}(t) < \frac{1}{2} X_{n_i}(s_i)\}, \\ \inf\{t > s_i | X_{n_i+2}(t) > 2X_{n_i}(s_i)\}, \\ s_i + \frac{2}{k_{n_i+1} \cdot a_{n_i}} \end{array} \right\}. \qquad (16)$$

The latter ensures that $t_i - s_i \leq \frac{2}{k_{n_i+1} \cdot a_{n_i}}$.

We now consider in separate cases which of the quantities attains the minimum in (16):

if $\quad X_{n_i}(t_i) = \frac{1}{2} X_{n_i}(s_i) \qquad$ then $\quad E_{n_i}(s_i) - E_{n_i}(t_i) \geq \frac{3}{4} a_{n_i}^2; \qquad (17)$

if $\quad X_{n_i+2}(t_i) = 2X_{n_i+2}(s_i) \quad$ then $\quad E_{n_i+1}(s_i) - E_{n_i+1}(t_i) \geq 3a_{n_i}^2; \qquad (18)$

if $\quad t_i = s_i + \dfrac{2}{k_{n_i+1} \cdot a_{n_i}} \qquad$ then $\quad E_{n_i}(s_i) - E_{n_i}(t_i) \geq 2^{-5-2\beta} a_{n_i}^2. \quad (19)$

The first and the second one are immediate, we prove the latter. If $t_i = s_i + \frac{2}{k_{n_i} \cdot a_{n_i}}$ then for all $s \in (s_i, t_i)$ we have $X_{n_i}(s) \geq \frac{1}{2} X_{n_i}(s_i)$ and $X_{n_i+2}(s) \leq 2X_{n_i+2}(s_i) \leq 2X_{n_i}(s_i)$, yielding

$$
\begin{aligned}
X'_{n_i+1}(s) &= k_{n_i} X^2_{n_i}(s) - k_{n_i+1} X_{n_i+1}(s) X_{n_i+2}(s) \\
&\geq \frac{1}{4} k_{n_i} X^2_{n_i}(s_i) - 2k_{n_i+1} X_{n_i+1}(s) X_{n_i}(s_i).
\end{aligned}
$$

Since $X_{n_i+1}(s_i) \geq 0$, we have (for all $t > s_i$)

$$
\begin{aligned}
X_{n_i+1}(t) &\geq \int_{s_i}^{t} \frac{1}{4} k_{n_i} X^2_{n_i}(s_i) e^{-2k_{n_i+1} X_{n_i}(s_i) \cdot (t-s)} ds \\
&\geq \frac{X_{n_i}(s_i)}{2^{3+\beta}} \left(1 - e^{-2k_{n_i+1} X_{n_i}(s_i) \cdot (t-s_i)} \right).
\end{aligned}
$$

This inequality allows to lower bound $E_{n_i}(s_i) - E_{n_i}(t_i)$ as follows

$$
\begin{aligned}
E_{n_i}(s_i) - E_{n_i}(t_i) &= \int_{s_i}^{t_i} 2k_{n_i} X^2_{n_i}(t) X_{n_i+1}(t) dt \\
&\geq \frac{k_{n_i} X^3_{n_i}(s_i)}{2^{4+\beta}} \int_{s_i}^{t_i} (1 - e^{-2k_{n_i+1} X_{n_i}(s_i) \cdot (t-s_i)}) dt \\
&\geq \frac{k_{n_i} X^3_{n_i}(s_i)}{2^{4+\beta}} \cdot \frac{1}{4} \cdot \frac{2}{k_{n_i+1} a_n} \geq \frac{X^2_{n_i}(s_i)}{2^{5+2\beta}}.
\end{aligned}
$$

This proves inequality (19).

Since E_n is non-increasing in t and $E_n(0) \leq ||x||^2_{l^2}$, from inequalities (17), (18) and (19), we deduce that for all $n \geq 1$

$$
\sharp\{i | n_i = n\} \leq \frac{||x||^2_{l^2}}{\frac{3}{4} a_n^2} + \frac{||x||^2_{l^2}}{3a_n^2} + \frac{||x||^2_{l^2}}{2^{-5-2\beta} a_n^2} \leq \frac{||x||^2_{l^2}}{a_n^2} 2^{6+2\beta}.
$$

Finally we are able to sum the measure of all intervals $[s_i, t_i)$:

$$
\begin{aligned}
\sum_i (t_i - s_i) &= \sum_{n \geq 1} \sum_{\{i | n_i = n\}} (t_i - s_i) \leq \sum_{n \geq 1} \frac{||x||^2_{l^2}}{a_n^2} 2^{6+2\beta} \cdot \frac{2}{k_{n+1} a_n} \\
&\leq 2^{7+\beta} ||x||^2_{l^2} \sum_n \frac{1}{k_n a_n^3}. \qquad \square
\end{aligned}
$$

Corollary 11. *Let $x = (x_n)_{n \geq 1} \in l^2$, with $x_n \geq 0$ for all n. Then for all $\alpha > \frac{1}{3}$ and for a.e. $t > 0$, the following inequality holds:*

$$
\sup_n n^{-\alpha} k_n^{\frac{1}{3}} X_n(t) < \infty.
$$

Proof. Simply apply Theorem 10 to the sequence $a_n := M n^\alpha k_n^{-\frac{1}{3}}$ and let M go to infinity. $\qquad \square$

The following is a more subtle consequence of Theorem 10 that will be needed to prove uniqueness, in the next section.

Corollary 12. *There exists a constant $c = c(\beta)$ such that, if $x = (x_n)_{n \geq 1} \in l^2$, $x_n \geq 0$ for all n, the following inequality holds for all $n \geq 1$ and $M > 0$.*

$$\mathcal{L}(X_n > M) := \mathcal{L}\{t \geq 0 | X_n(t) > M\} \leq \frac{c||x||_{l^2}^2}{k_n M^3}.$$

Proof. Take $L > M$ and define

$$a_i := \begin{cases} L & i < n \\ M & i \geq n. \end{cases}$$

Apply Theorem 10 to get:

$$\mathcal{L}(X_n > M) \leq 2^{8+\beta}||x||_{l^2}^2 \sum_i \frac{1}{k_i a_i^3}$$

$$\leq 2^{8+\beta}||x||_{l^2}^2 \left(\sum_{i=1}^{n-1} \frac{1}{k_i L^3} + \sum_{i=n}^{\infty} \frac{1}{k_i M^3} \right).$$

Taking the limit as L goes to infinity, we get

$$\mathcal{L}(X_n > M) \leq \frac{c(\beta)||x||_{l^2}^2}{k_n M^3}. \qquad \square$$

4   Uniqueness

The next step is to prove the uniqueness of solutions of system (1) with positive initial condition $x \in l^2$, for all $\beta > 1$. (The case $\beta \leq 1$ was already proved in [1].)

Theorem 13. *Let $x = (x_n)_{n \geq 1} \in l^2$ with $x_n \geq 0$ for all n. For all $\beta > 1$ there exists a unique weak solution X of (1) with initial condition x.*

Before starting the proof, we need some estimate of $\int_0^T X_N^3(t) dt$ for N large. Let N be an integer, $T > 0$ and $c = c(\beta)$ be the constant of Corollary 12. From the corollary, we deduce that for all $y \geq 0$,

$$\phi(y) := \mathcal{L}\{t \in [0, T] | X_N^3(t) > y\} \leq \min\left\{ \frac{c||x||_{l^2}^2}{k_N y}, T \right\}$$

Observe that $\phi(y) = 0$ for all $y > ||x||^3$, so that

$$\int_0^T X_N^3(t) dt = \int_0^\infty \phi(y) dy = \int_0^{||x||^3} \phi(y) dy$$

$$\leq \int_0^{||x||^3} \min\left\{ \frac{c||x||_{l^2}^2}{k_N y}, T \right\} dy \leq \frac{c||x||^2}{k_N T} T + \int_{\frac{c||x||^2}{k_N T}}^{||x||^3} \frac{c||x||_{l^2}^2}{k_N y} dy$$

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where we supposed that N is large enough that $K_N T > c||x||_{l^2}^{-1}$. By integrating, we obtain the following lemma.

Lemma 14. *Let $x = (x_n)_{n \geq 1} \in l^2$ with $x_n \geq 0$ for all n. Let $T > 0$, let $c = c(\beta)$ be the constant from Corollary 12. For all $N \geq 1$ such that $K_N T > c||x||_{l^2}^{-1}$ the following inequality holds*

$$\int_0^T X_N^3(t)dt \leq \frac{c(\beta)||x||^2}{k_N} \left(1 + \log \left(\frac{||x||T}{c(\beta)} \right) + N\beta \log(2) \right) \qquad (20)$$

This lemma shows that as N goes to infinity, the quantity $\int_0^T X_N^3(t)dt$ tends to zero at least as $\frac{N}{k_N}$. This is in accordance with the simple bound

$$\int_0^T X_N^2 X_{N+1}(t)dt \leq \frac{||x||^2}{k_N}$$

which is immediate by energy balance.

Proof of Theorem (13). The first part of the proof follows ideas in [1].

Let X and Y be two solution of system (1) with initial condition x. For all $n \geq 1$ let us define Z_n and W_n:

$$\begin{cases} Z_n := Y_n - X_n \\ W_n := Y_n + X_n \end{cases} \qquad \text{so that} \qquad \begin{cases} Y_n = \frac{W_n + Z_n}{2} \\ X_n = \frac{W_n - Z_n}{2} \end{cases} \qquad (21)$$

It is easy to verify that Z satysfies the following system

$$\begin{cases} Z_n' = k_{n-1} Z_{n-1} W_{n-1} - k_n \frac{Z_n W_{n+1} + W_n Z_{n+1}}{2}, \\ Z_n(0) = 0, \end{cases} \qquad n \geq 1, \quad t \geq 0$$

Since $Y_n(t) = X_n(t)$ if and only if $Z_n(t) = 0$, it will be sufficient to prove that $Z_n(t) \equiv 0$ for all $t \geq 0$ and $n \geq 1$. Let

$$\psi_N(t) := \sum_{n=1}^N \frac{Z_n^2}{2^n},$$

The functions $\psi_N(t)$ are non-negative, non-decreasing in N and such that $\psi_N(0) = 0$. We will prove that for all $t > 0$, $\lim_{N \to \infty} \psi_N(t) = 0$ concluding that $\psi_N(t) \equiv 0$ and $Z_n(t) \equiv 0$. We start by computing the derivative of ψ_N

$$\psi_N' = -\frac{k_N Z_N Z_{N+1} W_N}{2^N} - \sum_{n=1}^N \frac{k_n}{2^n} Z_n^2 W_{n+1}.$$

We observe that $W_n(t) = X_n(t) + Y_n(t)$ is non-negative and use (21) to get

$$\psi_N' \leq -2^{-N} k_N Z_N Z_{N+1} W_N \leq 2^{-N} k_N (Y_N^2 X_{N+1} + X_N^2 Y_{N+1})$$
$$\leq 2^{-N} k_N (Y_N^3 + X_{N+1}^3 + X_N^3 + Y_{N+1}^3).$$

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Since $\psi_N(0) = 0$ we deduce

$$\psi_N(t) \leq 2^{-N} k_N \int_0^t \left[Y_N^3(s) + X_{N+1}^3(s) + X_N^3(s) + Y_{N+1}^3(s) \right] ds$$

We can now apply Lemma 14 to get

$$\psi_N(t) \leq 2^{-N} k_N \frac{N}{k_N} c(\beta, t, ||x||)$$

where c is a constant not depending on N.

We conclude that $\lim_{N \to \infty} \psi_N(t) = 0$ for all $t > 0$. $\qquad\square$

5 The invariant region

In this section we work in the hypothesis $\beta \geq 1$ and $\sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} x_n < \infty$. We want to prove that $\sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} X_n(t)$ remains finite and uniformly bounded in t. (We already know from Corollary 11 that this quantity is finite for a.e. t.)

It is natural to consider the following change of variable: $Y_n(t) := k_n^{\frac{1}{3} - \frac{1}{3\beta}} X_n(t) = 2^{\frac{\beta-1}{3}n} X_n(t)$. From (1) we obtain that $(Y_n)_{n \geq 1}$ solves

$$\begin{cases} Y_n' = 2^{\frac{2\beta+1}{3}n - \frac{\beta+2}{3}} \left(Y_{n-1}^2 - 2 Y_n Y_{n+1} \right), \\ Y_n(0) = y_n, \end{cases} \qquad n \geq 1, \quad t \geq 0 \qquad (22)$$

with $y_n := 2^{\frac{\beta-1}{3}n} x_n$.

For technical reasons we consider a finite dimensional (truncated) version of the equations for Y. For every $N \geq 1$ let $(Y_n^{(N)})_{1 \leq n \leq N}$ be the solution to

$$\begin{cases} \frac{d}{dt} Y_n^{(N)} = 2^{\frac{2\beta+1}{3}n - \frac{\beta+2}{3}} \left[\left(Y_{n-1}^{(N)} \right)^2 - 2 Y_n^{(N)} Y_{n+1}^{(N)} \right], \\ Y_n^{(N)}(0) = y_n, \end{cases} \qquad (23)$$

for $n = 1, \ldots, N$, where for the sake of simplicity we have set $Y_0^{(N)} = 0$ and $Y_{N+1}^{(N)} = Y_N^{(N)}$, so to avoid writing the border equations in a different form. Let us now introduce the region A of $\mathbb{R}^2$ that will be invariant for the vectors $(Y_n^{(N)}, Y_{n+1}^{(N)})$,

$$A := \{(x, y) \in [0, 1]^2 : h(x) \leq y \leq g(x)\},$$

where the functions h and g that provide the bounds of A are defined as

$$g(x) = mx + \theta, \qquad h(x) = c \left(\frac{x - \delta}{1 - \delta} \right)^3$$

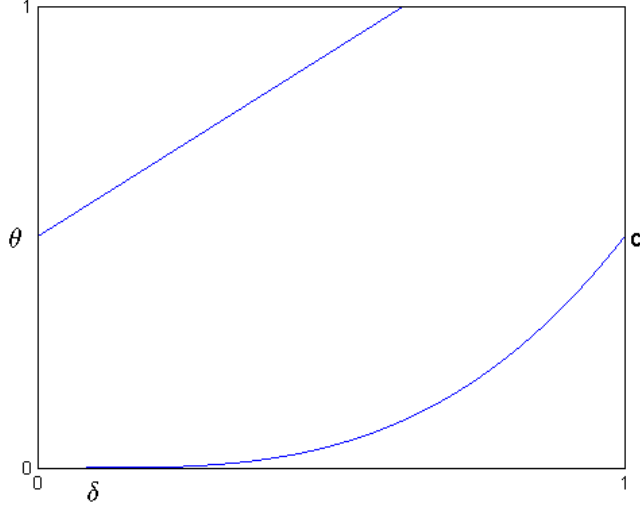so that $g \leq 1$ on $[0, \frac{1-\theta}{m}]$ and $h \geq 0$ on $[\delta, 1]$.

Figure 1: Invariant Region

Theorem 15. *For $\delta = \frac{1}{12}$, $c = \frac{1}{2}$, $\theta = \frac{1}{2}$, $m = \frac{4}{5}$ and for all $\beta \geq 1$ the region A is an invariant region, that is if (y_n, y_{n+1}) belongs to A for all $n \leq N - 1$ then $(Y_n^N(t), Y_{n+1}^N(t))$ belongs to A for all $n \leq N - 1$ and $t \geq 0$.*

Proof. For notation semplicity we drop the superscript N along the proof. We follow and improve ideas from [5]. Consider the set

$$B := \{z \in [0,1]^N : (z_n, z_{n+1}) \in A, n = 1, 2, \ldots, N - 1\}$$

We want to prove that $Y \in B$ for all times, knowing that this is true at time zero. This will follow from the fact that $\frac{d}{dt}Y$ points inward on all the border of B.

By the definition of B, a point $Y = (Y_1, \ldots, Y_N)$ is on the border if (at least) one of the following constraints holds with equality

1. $h(Y_n) \leq Y_{n+1}$, for all $n \leq N - 1$;

2. $Y_{n+1} \leq g(Y_n)$, for all $n \leq N - 1$;

3. $0 \leq Y_n \leq 1$, for all $n \leq N$.

For the first two cases, we must check that for $n = 1, 2, \ldots, N - 1$, the derivative in time of the vector (Y_n, Y_{n+1}) points inward on the border of A when $Y \in B$. Steps 1 and 2 below deal with these cases.

In the third case, the positivity is trivial by Proposition 4, while the other constraint can be easily checked in dimension 1, as we do in step 3.

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Step 1. On the constraint $Y_{n+1} = h(Y_n)$.

Let us start studing the border 'along h' that is $Y_n \in [\delta, 1]$ and $Y_{n+1} = h(Y_n)$. The inward normal is given by $(-h'(Y_n), 1)$. From equation (23) we obtain

$$\begin{pmatrix} Y_n' \\ Y_{n+1}' \end{pmatrix} \sim \begin{pmatrix} Y_{n-1}^2 - 2Y_n Y_{n+1} \\ 2^{\frac{2\beta+1}{3}} (Y_n^2 - 2Y_{n+1} Y_{n+2}) \end{pmatrix}$$

where $\sim$ means that the two vectors have the same direction and sense.

Consider the scalar product:

$$\langle (-h'(Y_n), 1), (Y_n', Y_{n+1}') \rangle =$$
$$- h'(Y_n) \cdot (Y_{n-1}^2 - 2Y_n Y_{n+1}) + 2^{\frac{2\beta+1}{3}} (Y_n^2 - 2Y_{n+1} Y_{n+2})$$
$$\geq - h'(Y_n) \cdot (1 - 2Y_n h(Y_n)) + 2^{\frac{2\beta+1}{3}} (Y_n^2 - 2h(Y_n) g(h(Y_n))) \quad (24)$$

Where the inequality comes from the fact that $(Y_1, \dots, Y_N) \in B$ and $Y_{N+1} = Y_N$.

Since we want the right-hand side of (24) to be positive for arbitrarily large β, first of all we must check that $\Phi_1(x) := x^2 - 2h(x)g(h(x))$ is positive for all $x \in [\delta, 1]$ and this is not difficult to verify, since Φ_1 is just a real polinomial of degree 6. (See it plotted in Figure 5-a.)

As $\Phi_1 > 0$, we shall lower bound the right-hand side of (24) by setting $\beta = 1$, yielding

$$\langle (-h'(Y_n), 1), (Y_n', Y_{n+1}') \rangle \geq -h'(Y_n) \cdot (1 - 2Y_n h(Y_n)) + 2\Phi_1(Y_n) =: \Phi_2(Y_n)$$

where again Φ_2 is a real polinomial of degree 6 not depending on β and positive on all $[\delta, 1]$. (See Figure 5-b.)

Step 2. On the constraint $Y_{n+1} = g(Y_n)$.

Let us now turn to the border 'along g' that is $Y_n \in \left[0, \frac{1-\theta}{m}\right]$ and $Y_{n+1} = g(Y_n)$. The inward normal is given by $(m, -1)$, whereas the scalar product is given by:

$$\langle (m, -1), (Y_n', Y_{n+1}') \rangle =$$
$$m \left(Y_{n-1}^2 - 2Y_n Y_{n+1} \right) - 2^{\frac{2\beta+1}{3}} (Y_n^2 - 2Y_{n+1} Y_{n+2})$$
$$\geq m \left(0 - 2Y_n g(Y_n) \right) - 2^{\frac{2\beta+1}{3}} (Y_n^2 - 2g(Y_n) h(g(Y_n))) \quad (25)$$

In analogy with the previous case, we define $\Phi_3(x) := x^2 - 2g(x)h(g(x))$ and check that this degree 4 polynomial is negative for $x \in \left[0, \frac{1-\theta}{m}\right]$. We can then lower bound (25) by setting $\beta = 1$, yielding

$$\langle (m, -1), (Y_n', Y_{n+1}') \rangle \geq -2mY_n g(Y_n) - 2\Phi_3(Y_n) =: \Phi_4(Y_n)$$

One can check that Φ_4 is a degree 4 polynomial positive on the interval $\left[0, \frac{1-\theta}{m}\right]$. (See Figure 5-c and 5-d.)
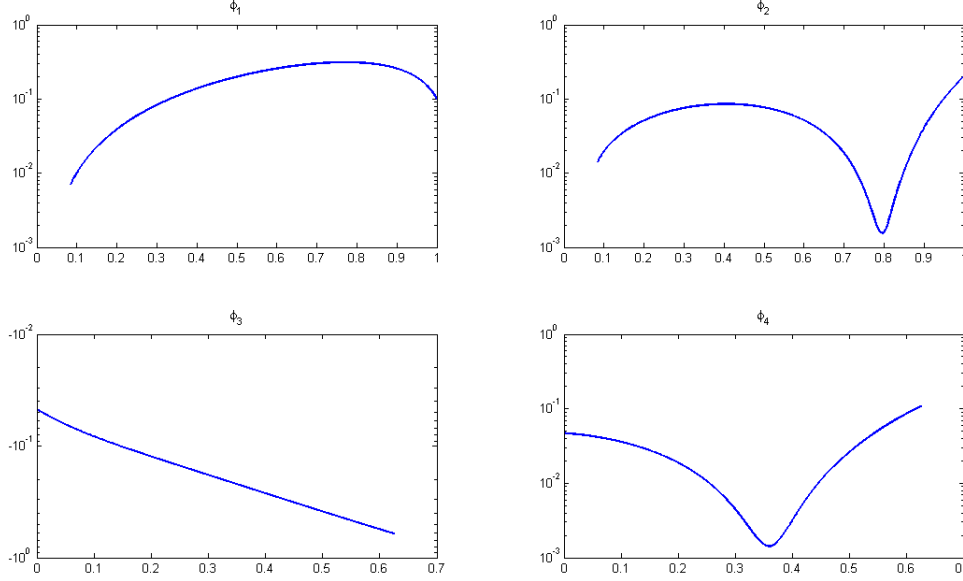
Figure 2: Control functions

Step 3. On the constraint $Y_n = 1$.
We must prove that $Y'_n \leq 0$ when $Y_n = 1$ and $Y \in B$, but these hypothesis imply that $Y_{n+1} \geq h(1) = c$, so that

$$Y'_n = 2^{\frac{2\beta+1}{3}n - \frac{\beta+2}{3}} \left(Y_{n-1}^2 - 2Y_n Y_{n+1} \right) \leq 2^{\frac{2\beta+1}{3}n - \frac{\beta+2}{3}} (1 - 2c) = 0$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With standard techniques, making the limit on N in the above theorem it is possible to obtain the following:

Corollary 16. *Let $\beta \geq 1$ and let $x = (x_n)_{n \geq 1}$ be such that $x_n \geq 0$ for all $n \geq 1$ and $\sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} x_n < \infty$. If X is the weak solution, then*

$$\sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} X_n(t) \leq 12 \sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} x_n$$

for all $t \geq 0$.

Proof. Remember that equation (3) defines a rescaling which is still a solution of the original system. Let $L := \sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} x_n$ and $\delta = 1/12$ as in Theorem 15. Then $W(t) := \frac{\delta}{L} X(\frac{\delta}{L} t)$ is the unique solution of system (1) with initial condition $\frac{\delta}{L} x$. As in the beginning of this section, define $Y = (Y_n)_{n \geq 1}$, with $Y_n := k_n^{\frac{1}{3} - \frac{1}{3\beta}} W_n$. Uniqueness for system (22) follows from uniqueness

for system (1), so Y is the unique solution of the former, with initial condition $y = (y_n)_{n \geq 1}$, defined by $y_n = \frac{\delta}{L} k_n^{\frac{1}{3} - \frac{1}{3\beta}} x_n \in [0, \delta]$. Since $[0, \delta]^2 \subset A$, Theorem 15 applies and thanks to uniqueness, it is standard to prove that $Y^{(N)}$ converges to Y, so we get $Y_n(t) \leq 1$ uniformly in n and t, which is what we had to prove. $\qquad\square$

Theorem 17. *Let X be the unique solution of system* (1) *with initial condition x with non-negative components. Then there exists a costant $c(\beta) > 0$ such that the following inequality holds for all $t > 0$:*

$$\sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} X_n(t) \leq c(\beta) \|x\|^{\frac{2}{3}} t^{-1/3}.$$

Proof. Let $\psi(t) := \sup_n k_n^{\frac{1}{3} - \frac{1}{3\beta}} X_n(t)$. By Corollary 16 we have,

$$\psi(t) \leq 12\psi(s) \qquad \forall \ s \in [0, t].$$

To complete the proof it is sufficient to fix $c(\beta)$ in such a way that there exists $s \in [0, t]$ such that

$$12\psi(s) \leq c(\beta)\|x\|^{\frac{2}{3}} t^{-1/3} \tag{26}$$

Let us consider Theorem 10. Letting $a_n = M/r_n$, the thesis rewrites

$$\mathcal{L}\{s > 0 | \sup_{n \geq 1}\{r_n X_n(s)\} > M\} \leq 2^{8+\beta}\|x\|^2 \sum_{n \geq 1} \frac{r_n^3}{k_n M^3},$$

Now let

$$r_n = 12 k_n^{\frac{1}{3} - \frac{1}{3\beta}}, \qquad M = c(\beta)\|x\|^{\frac{2}{3}} t^{-1/3}, \qquad c(\beta) > 12 \cdot 2^{\frac{8+\beta}{3}},$$

we get

$$\mathcal{L}\Big\{s > 0 \Big| 12\psi(s) > c(\beta)\|x\|^{\frac{2}{3}} t^{-1/3}\Big\} \leq 12^3 \frac{2^{8+\beta}}{c(\beta)^3} t < t,$$

hence there exists $s \in [0, t]$ such that inequality (26) holds. $\qquad\square$

References

[1] David Barbato, Franco Flandoli, and Francesco Morandin. A theorem of uniqueness for an inviscid dyadic model. *C. R. Math. Acad. Sci. Paris*, 348(9-10):525–528, 2010.

[2] David Barbato, Franco Flandoli, and Francesco Morandin. Uniqueness for a stochastic inviscid dyadic model. *Proc. Amer. Math. Soc.*, 138(7):2607–2617, 2010.

[3] David Barbato, Franco Flandoli, and Francesco Morandin. Anomalous dissipation in a stochastic inviscid dyadic model. *Annals of Applied Probability*, 21(6):2424–2446, 2011.

[4] David Barbato, Franco Flandoli, and Francesco Morandin. Energy dissipation and self-similar solutions for an unforced inviscid dyadic model. *Trans. Amer. Math. Soc.*, 363(4):1925–1946, 2011.

[5] David Barbato, Francesco Morandin, and Marco Romito. Smooth solutions for the dyadic model. *Nonlinearity*, 24(11):3083, 2011.

[6] L. Biferale. Shell models of energy cascade in turbulence. *Annu. Rev. Fluid Mech.*, 35:441–468, 2003.

[7] Alexey Cheskidov. Blow-up in finite time for the dyadic model of the Navier-Stokes equations. *Trans. Amer. Math. Soc.*, 360(10):5101–5120, 2008.

[8] Alexey Cheskidov, Susan Friedlander, and Nataša Pavlović. An inviscid dyadic model of turbulence: the global attractor. *Discrete Contin. Dyn. Syst.*, 26(3):781–794, 2010.

[9] V. N. Desnianskii and E. A. Novikov. Simulation of cascade processes in turbulent flows. *Prikladnaia Matematika i Mekhanika*, 38:507–513, 1974.

[10] Jens Eggers and Siegfried Grossman. Anomalous turbulent velocity scaling from the navier-stokes equation. *Physics Letters A*, 156:444–449, 1991.

[11] Nets Hawk Katz and Nataša Pavlović. Finite time blow-up for a dyadic model of the Euler equations. *Trans. Amer. Math. Soc.*, 357(2):695–708 (electronic), 2005.

[12] Alexander Kiselev and Andrej Zlatoš. On discrete models of the Euler equation. *Int. Math. Res. Not.*, (38):2315–2339, 2005.

[13] Fabian Waleffe. On some dyadic models of the Euler equations. *Proc. Amer. Math. Soc.*, 134(10):2913–2922 (electronic), 2006.